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Nonlinear Jaynes–Cummings Model of Atom–Field Interaction

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Interaction of a two-level atom with a single mode of electromagnetic field including Kerr nonlinearity for the field and intensity-dependent atom-field coupling is discussed. The Hamiltonian for the atom-field system is written in terms of the generators of a closed algebra, which has SU(1,1) and Heisenberg–Weyl algebras as limiting cases. Eigenstates and eigenvalues of the Hamiltonian are constructed. With the field being in a coherent state initially, the dynamical behavior of atomic inversion, field statistics, and uncertainties in the field quadratures are studied. Appearance of nonclassical features during the evolution of the field is shown. Further, we explore the overlap of initial and time-evolved field states.

KEY WORDS: Jaynes–Cummings model; atomic inversion; revivals; squeezing.

1. INTRODUCTION

Interaction of a two-level atom with a single mode of electromagnetic field is the simplest problem in matter–radiation coupling. A model for the interaction, introduced by Jaynes and Cummings (1963), treats the atom as a dipole placed in an external field. The Jaynes–Cummings (JC) model has provided a lot of impetus for theoretical explorations and experimental verifications (Brune *et al*., 1987; Gentile *et al*., 1989; Kozierowski and Chumakov, 2001; Shore and Knight, 1993; Walther, 1993). The Hamiltonian for the model is

$$H_{\mathrm{JC}} = H_0 + g(\hat{a}^\dagger \sigma_- + \hat{a}\sigma_+). \tag{1}$$

Here H_0, the Hamiltonian for the atom and field in the absence of interaction, is the sum of energy operators for the field and atom given by $\omega \hat{a}^\dagger \hat{a} + \frac{1}{2}\nu\sigma_z$. The atomic transition frequency is ν, the field frequency is ω, and g is the coupling constant. We denote the creation operator by $\hat{a}^\dagger$, annihilation operator by $\hat{a}$ for the field quanta, and their action on the basis states of the harmonic

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oscillator are

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle, \tag{2}$$

$$\hat{a}|0\rangle = 0, \tag{3}$$

$$\hat{a}^{\dagger}|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{4}$$

The atom has two levels, $|g\rangle$ and $|e\rangle$, the ground and excited states. The operators σ_z, σ_+, and σ_- act on the states as given below:

$$\sigma_z|g\rangle = -|g\rangle, \tag{5}$$

$$\sigma_z|e\rangle = |e\rangle, \tag{6}$$

$$\sigma_{\pm}|g\rangle = \frac{1 \pm 1}{2}|e\rangle, \tag{7}$$

$$\sigma_{\pm}|e\rangle = \frac{1 \mp 1}{2}|g\rangle. \tag{8}$$

It is noted that the operator $\frac{\nu}{2}\sigma_z$ is the energy operator for the two-level atom.

Atomic inversion, defined as the difference in probabilities for the atom to be in the excited and ground states, as predicted by JC model is a sum of quasiperiodic functions with incommensurate frequencies. The model predicts collapses and revivals and ringing revivals in the time-development of atomic inversion (Cummings, 1965; Meystre and Quattropani, 1975; Stenholm, 1973). Analytical evaluation of the time required for the first collapse and subsequent revival was done in Eberly *et al*., 1983. If the photon-number distribution is osicllatory, as in the case of squeezed vacuum, it is echoed in the "ringing revival" structures of the atomic inversion (Venkata Satyanarayna *et al*., 1989). The revival phenomenon is entirely quantal, and hence the model is very important in experimental verification of the predictions of quantum theory. Although the model is based on simple assumptions regarding the matter–radiation interaction, it has been extensively used to study a variety of phenomena like trapping of atoms (Li *et al*., 1998), electromagnetically induced transparency and enhancement of refractive index (Scully and Zubairy, 1997), mechanical action of light on atoms (Kazantsev *et al*., 1990), etc. Further, the model has been generalized in many ways. We list the respective Hamiltonian for some of the extensions:

1) Buck-Sukumar model (Buck and Sukumar, 1981)

$$H_{\mathrm{IC}} = H_0 + g(\sqrt{\hat{a}^{\dagger}\hat{a}}\hat{a}\sigma_+ + \hat{a}^{\dagger}\sqrt{\hat{a}^{\dagger}\hat{a}}\sigma_-) \tag{9}$$

With this particular form of intensity-dependent coupling the atomic-inversion is a sum of periodic functions, to be precise, the discrete Fourier transform of the photon number distribution. This model is very interesting as it can be written as a combination of generators of SU(1,1) algebra

(Buzek, 1989). Generalization to two-photon case, where two photons are absorbed or emitted in atom–field interactions, has been done. Once again the SU(1,1) algebraic structure in the model is used to solve the problem (Gerry, 1988).

2) Kerr nonlinearity (Gora and Jedrzejek, 1992; Joshi and Puri, 1992; Werner and Risken, 1991)

$$H_{\text{Kerr}} = H_{\text{JC}} + \chi \hat{a}^{\dagger 2} \hat{a}^2 \tag{10}$$

This is an effective Hamiltonian for a system in which the electromagnetic field mode is excited in a Kerr medium. The medium is modeled as an anharmonic oscillator (Agarwal and Puri, 1989; Yurke and Stoler, 1986).

3) Dicke–Tavis–Cummings model (Dicke, 1954; Tavis and Cummings, 1968) In this model, the interaction between field and a group of two-level atoms is considered and the Hamiltonian is

$$H_{\text{DTC}} = \omega \hat{a}^\dagger \hat{a} + \frac{\nu}{2} \sum_i \sigma_{i,z} + \text{Interaction part.} \tag{11}$$

Generalization of the model to include multimode field configurations has been studied in Abdel-Hafez and Ahmad (1987).

4) Nonlinear Jaynes–Cummings model

On including the motion of atom in the external field, the coupling is made position dependent. This offers enormous possibilities to tailor the form of atom–field interaction. The general form for the Hamiltonian is

$$H_{\text{NL}} = H_0 + g(f(\hat{a}^\dagger \hat{a}) a^m + \text{adjoint}) \tag{12}$$

Here $f(\hat{a}^\dagger \hat{a})$ is an operator-valued function of the number operator $\hat{a}^\dagger \hat{a}$ and m is an integer. Supersymmetric technique to solve multiphoton nonlinear Jaynes–Cummings model are explored in Song and Fan (2002).

One way of realizing nonlinear JC hamiltonians is to consider atomic systems with vibrational sidebands (Lie and Wang, 1996; Vogel and de Matos Filho, 1995). The atom interacts with two external fields which are treated classically. The center-of-mass motion (also referred as external degree of freedom) of the atom trapped in a cavity is coupled to its internal degrees of freedom, namely, the vibrational sidebands. The spatial structure of the cavity field determines the form for the function $f(\hat{a}^\dagger \hat{a})$. By proper choice of cavity mode structure, it is possible to design arbitrary atom–field couplings.

Another way to arrive at atom–field coupling, which has polynomial dependence on the photon number, is to use many lasers with different phases and Rabi frequencies to interact with a trapped ion with vibrational sidebands. The number of lasers required is same as the order of the polynomial. It has been shown that once the function $f(n)$, considered to be a polynomial, is specified, the required phases and the Rabi frequencies of the external lasers are determined (de Matos

Filo and Vogel, 1998). A more realistic case is to consider the interaction of the trapped atom with environment. It has been shown that by varying laser frequencies and their intensities it is possible to engineer the atom–environment coupling too (Poyatos *et al*., 1996). The coupling comes about due to the absorption of a laser photon and subsequent spontaneous emission.

With the technology advancement in trapping and availability of lasers, tailoring the atom–field interaction is possible. In the case of laser cooled, trapped ions the effect of dissipation can be made negligible for sufficiently long times so that experimentation is possible. In this context, we study the dynamics of a two-level atom interacting with a single mode of electromagnetic field. The interaction is governed by the Hamiltonian

$$H = \omega\left[\hat{a}^\dagger\hat{a} + \frac{r}{2}\sigma_z\right] + \chi\hat{a}^{\dagger 2}\hat{a}^2 + g\omega(\sqrt{1+k\hat{a}^\dagger\hat{a}}\hat{a}\sigma_+ + \hat{a}^\dagger\sqrt{1+k\hat{a}^\dagger\hat{a}}\sigma_-). \tag{13}$$

This Hamiltonian is an example for nonlinear JC model including Kerr term. The time evolution of atom–field system governed by the Hamiltonian is exactly solvable. Note that we have scaled the coupling constant g by ω. The parameter r is $\frac{\nu}{\omega}$ and the coupling constant for the Kerr term is $\chi = k\omega$. This will simplify the expressions we derive in sequel. In particular, we set $\omega = 1$ which amounts to studying a new Hamiltonian $\frac{H}{\omega}$. The speciality of the Hamiltonian H is that it becomes H_{JC} when the parameter k is set equal to zero. Further, the usual Holstein–Primakoff realization is obtained when $k = 1$, so that the interaction is approximately given by $g(\sqrt{\hat{a}^\dagger\hat{a}}\hat{a}_+ + \text{adjoint})$, which is same as the interaction studied in Buzek (1989). Yet another form of interaction occurs when $k \ll 1$. If the photon number distribution is such that $kn \ll 1$ for all n under the peak of the distribution, then Eq. (13) leads to

$$H \to \omega[\hat{a}^\dagger\hat{a} + k\hat{a}^{\dagger 2}\hat{a}^2 + g[(1 + k\hat{a}^\dagger\hat{a}/2)\hat{a}\sigma_+ + \text{adjoint}]]. \tag{14}$$

The coupling given here, and the one including the next higher term proportional to $\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}$, are realizable with four lasers interacting with a trapped two-level atom (de Matos Filo and Vogel, 1998). Further, when $k \ll 1$ so that we can neglect kn in comparison to unity but retain kn^2, we arrive at H_{JC} with an additional Kerr term. This system has been studied in Bernat and Jex (1992). Many well-studied systems are thus special cases of the Hamiltonian considered for discussion in the present paper. The organization of the paper is as follows. In Section 2 we study the algebraic aspects of the Hamiltonian and construct the eigenvectors and eigenvalues. Section 3 is devoted to study the atomic inversion, and approximate expressions are obtained for first collapse and revival periods when the field is in a coherent state. The time-development of field statistics is discussed in Section 4 and the results are summarized in Section 5.

2. MODEL HAMILTONIAN AND ITS PROPERTIES

In this section we study the algebraic aspects of the generalized Hamiltonian given in Eq. (13). We introduce a set of operators which are closed under commutation. This algebraic structure is exploited to determine the time-evolution operator to evolve the initial state of the atom–field system. The eigenvalues and the corresponding eigenvectors are constructed.

2.1. Eigenvalues and Eigenvectors

The Hamiltonian given in Eq. (13) is written as

$$H = \omega K_+ K_- + \frac{\nu}{2}\sigma_z + g(K_+\sigma_- + K_-\sigma_+), \tag{15}$$

wherein we have set $K_- = \sqrt{1 + k\hat{a}^\dagger\hat{a}}\,\hat{a}$ and $K_+ = \hat{a}^\dagger\sqrt{1 + k\hat{a}^\dagger\hat{a}}$. Further, we assume that k is nonnegative and restricted to take values less than or equal to unity. Formally, the Hamiltonian H has the same structure as $H_{\rm JC}$ with $\hat{a}$ and $\hat{a}^\dagger$ replaced by K_- and K_+. However, the former corresponds to Kerr-type medium with intensity-dependent coupling for atom–field interaction. The difference is very clear in the commutation relations among the operators. From the realization of K_- and K_+ in terms of $\hat{a}^\dagger$ and $\hat{a}$, we arrive at

$$\begin{aligned} [K_-, K_+] &= 2K_0, \\ [K_0, K_\pm] &= \pm k K_\pm. \end{aligned} \tag{16}$$

The operator K_0 is $k\hat{a}^\dagger\hat{a} + \frac{1}{2}$. Thus, the operators K_-, K_+, and K_0 form a closed algebra. It is worth noting that the commutation relations define the SU(1,1) algebra when $k = 1$. On the other hand, to get the well-known Heisenberg–Weyl algebra generated by $\hat{a}^\dagger$, $\hat{a}$, and the identity I we set $k = 0$. Two different algebras are realized depending on the value of k and hence the algebra of K_-, K_+, and K_0 is said to be an "interpolating algebra." An invariant operator, which commutes with $K_\pm$ and K_0, for the algebra is given by $K_0^2 - (k/2)(K_-K_+ + K_+K_-)$. The coherent states corresponding to this algebra and their Hilbert space properties are known (Sivakumar, 2002).

The atom–field evolution is studied in the space of $|e, n\rangle$ and $|g, n\rangle$, where $n = 0, 1, 2, \ldots$. The state $|e, n\rangle$ means that the atom is in the excited state $|e\rangle$ and the field in the nth excited state $|n\rangle$. The states $|e, n\rangle$ and $|g, n\rangle$ are eigenstates of $\omega K_+K_- + \frac{\nu}{2}\sigma_z$ and the respective eigenvalues are $\varepsilon_{e,n} = (n + kn^2 - kn)\omega + \frac{\nu}{2}$ and $\varepsilon_{g,n} = (n + kn^2 - kn)\omega - \frac{\nu}{2}$. The Hamiltonian H admits a constant of motion N such that the commutator $[N, H]$ vanishes. Explicitly, $N = K_+K_- + 2K_0\sigma_+\sigma_-$. Note that when $k = 0$, N becomes $\hat{a}^\dagger\hat{a} + \sigma_+\sigma_-$, the constant of motion for $H_{\rm JC}$.

The interaction part of H is such that the state $|e, n\rangle$ is taken to $|g, n + 1\rangle$ and vice versa, during the evolution of the atom–field system. Thus, the entire Hilbert

space is split into subspaces spanned by $|e, n\rangle$ and $|g, n+1\rangle$ and the dynamics confined to individual subspaces. In one such subspace, specified by the value of n, the Hamiltonian matrix is

$$H = \begin{pmatrix} \left(n + \frac{1}{2} + kn^2 - kn\right)\omega + \Delta/2 & g\sqrt{(1+kn)(1+n)} \\ g\sqrt{(1+kn)(1+n)} & \left(n + \frac{1}{2} + kn^2 + kn\right)\omega - \Delta/2 \end{pmatrix}. \tag{17}$$

The detuning parameter Δ is $(r-1)\omega$. The eigenvalues of the Hamiltonian are

$$E_{\pm,n} = \left(kn^2 + n + \frac{1}{2}\right)\omega \pm \frac{1}{2}\sqrt{(\Delta - 2k\omega n)^2 + 4g^2\omega^2(1+kn)(1+n)}. \tag{18}$$

and the corresponding eigenvectors are

$$|+, n\rangle = \cos\theta_n |e, n\rangle + \sin\theta_n |g, n+1\rangle, \tag{19}$$

$$|-, n\rangle = \sin\theta_n |e, n\rangle - \cos\theta_n |g, n+1\rangle. \tag{20}$$

The expansion coefficients are

$$\cos\theta_n = \frac{2g\omega\sqrt{(1+n)(1+kn)}}{\sqrt{(\Omega_n - \Delta_n)^2 + 4g^2\omega^2(1+n)(1+kn)}} \tag{21}$$

$$\sin\theta_n = \frac{\Omega_n - \Delta_n}{\sqrt{(\Omega_n - \Delta_n)^2 + 4g^2\omega^2(1+n)(1+kn)}}, \tag{22}$$

in which we have set $\Delta_n = \Delta - 2kn\omega$ and $\Omega_n = \sqrt{\Delta_n^2 + 4g^2\omega^2(1+n)(1+kn)}$.

The energy difference between the levels $E_{+,n}$ and $E_{-,n}$ is $\sqrt{\Delta_n + 4g^2\omega^2(1+kn)(1+n)}$. The minimum of the separation occurs when Δ equals $2kn\omega$ and the corresponding difference is $2g\omega\sqrt{(1+kn)(1+n)}$. In Fig. 1 we have plotted the energy eigenvalues E_+ and E_- as a function of Δ. The dashed lines represent the eigenvalues when $g = 0$, i.e., $\varepsilon_{e(g),n}$. In this case the eigenvalues cross each other as Δ increases from negative to positive values. The continuous lines represent the energy eigenvalues for $g = 10^{-3}$. The diverging eigenvalue separation beyond the minimum separation indicates "level repulsion" in the eigenvalues of the dressed atom. The effect of nonzero k is to shift the value of Δ at which the minimum separation or the crossing occurs. If $k = 0$, the minimum as well as the crossing occur at $\Delta = 0$.

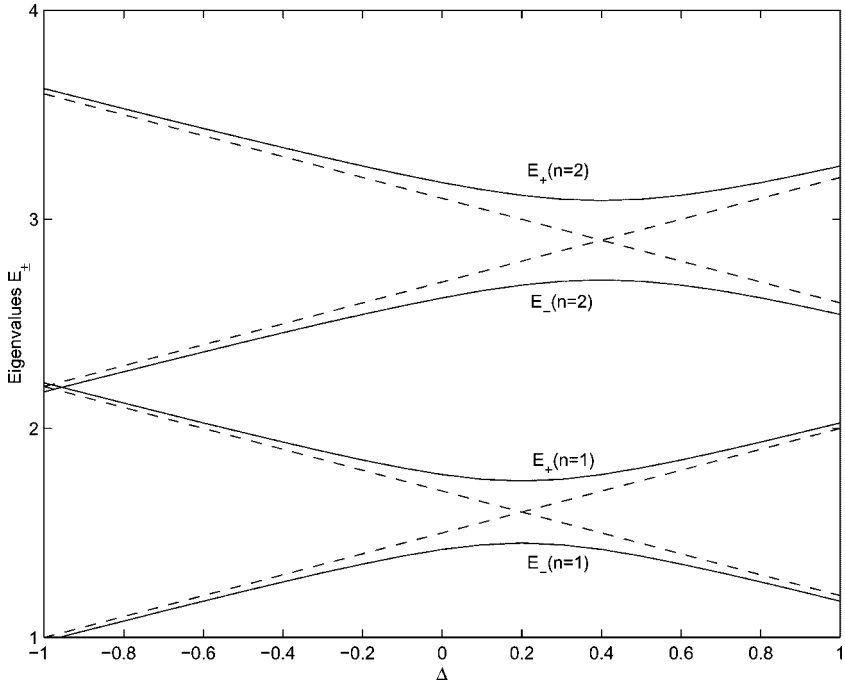

Fig. 1. Dependence of eigenvalues E_+ and E_- on detuning Δ. The continuous curve corresponds to $g = 0.1$ and the dashed curve corresponds to $g = 0$. Here $k = .1$ and $\bar{n} = 30$. The dashed curves with positive and negative slopes correspond respectively to $\varepsilon_{e,n}$ and $\varepsilon_{g,n}$. Lower part of the figure is for $n = 1$ and upper part for $n = 2$.

2.2. Evolution of Atom–Field State

To understand the dynamics of the atom–field system, we solve for the state of the system in interaction picture, where the evolution equation is

$$i\frac{\partial|\psi\rangle}{\partial t} = \tilde{V}|\psi\rangle. \tag{23}$$

Here $\tilde{V}$ is the transformed interaction given by

$$\tilde{V} = g\exp\left[it\left(\omega K_+K_- + \frac{\nu}{2}\sigma_z\right)\right](\sigma_-K_+ + \sigma_+K_-)\exp\left[-it\left(\omega K_+K_- + \frac{\nu}{2}\sigma_z\right)\right]. \tag{24}$$

The effect of the transformation on the interaction term is obtained from the following results:

$$\exp(it\omega K_+K_-)K_+\exp(-it\omega K_+K_-) = K_+\exp(it\omega K_0), \tag{25}$$

$$\exp\left(\frac{itv}{2}\sigma_z\right)\sigma_-\exp\left(-\frac{itv}{2}\sigma_z\right)=\exp(-itv)\sigma_-, \tag{26}$$

and their adjoints. Note that the ordering of operators should be maintained in the rhs of the first of the results. Using these relations, the interaction picture Hamiltonian is written as

$$\tilde{V}=g(K_+\sigma_-\exp[it(2\omega K_0-v)]+\text{adjoint}. \tag{27}$$

At any time t, let the state of the atom–field system be represented as

$$|\psi(t)\rangle=\sum_{n=0}^{\infty}C_{e,n}(t)|e,n\rangle+C_{g,n}(t)|g,n\rangle. \tag{28}$$

The coefficients $C_{e,n(t)}$ and $C_{g,n(t)}$, determined in terms of their initial values by the evolution equation Eq. (23), are

$$\exp\left(\frac{-i\Delta_n t}{2}\right)C_{e,n}(t)=\left[\cos\left(\frac{\Omega_n t}{2}\right)-\frac{i\Delta_n}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right)\right]C_{e,n}(0) -\frac{2ig\omega\sqrt{(n+1)(1+kn)}}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right)C_{g,n+1}(0) \tag{29}$$

$$\exp\left(\frac{i\Delta_n t}{2}\right)C_{g,n+1}(t)=\left[\cos\left(\frac{\Omega_n t}{2}\right)-\frac{i\Delta_n}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right)\right]C_{g,n+1}(0) -\frac{2ig\omega\sqrt{(n+1)(1+kn)}}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right)C_{e,n}(0). \tag{30}$$

The Rabi frequency Ω_n is $\sqrt{\Delta_n^2+4g^2\omega^2(1+kn)(1+n)}$. The dependence of Ωn on n such that there is a minimum value for the Rabi frequency when n satisfies $\Delta=2kn\omega+g^2\omega^2(1+k+2kn)(k\omega)^{-1}$, provided $k\neq 0$. The variation of Ω_n with respect to n is shown in Fig. 2. In the case of H_{JC}, the Rabi frequency varies linearly with n and hence there is no minimum. The existence of a minimum Rabi frequency has important consequences for the dynamics of atomic inversion, squeezing, photon statistics, etc. and they are discussed in the following sections.

Let the Rabi frequency attain its minimum for some specific value of n, denoted by $\bar{n}$. Therefore, we have

$$\Delta=\Delta_c=2k\omega\bar{n}+\frac{g^2\omega^2(1+k+\bar{n})}{k\omega}. \tag{31}$$

The n dependence of Rabi frequency, for values of n close to $\bar{n}$, is obtained by Taylor expanding Ω_n around $\bar{n}$, up to second order. The resultant expression is

$$\Omega_n=\Omega_{\bar{n}}+\frac{2(n-\bar{n})^2(k^2\omega^2+kg^2\omega^2)}{\Omega_{\bar{n}}} \tag{32}$$

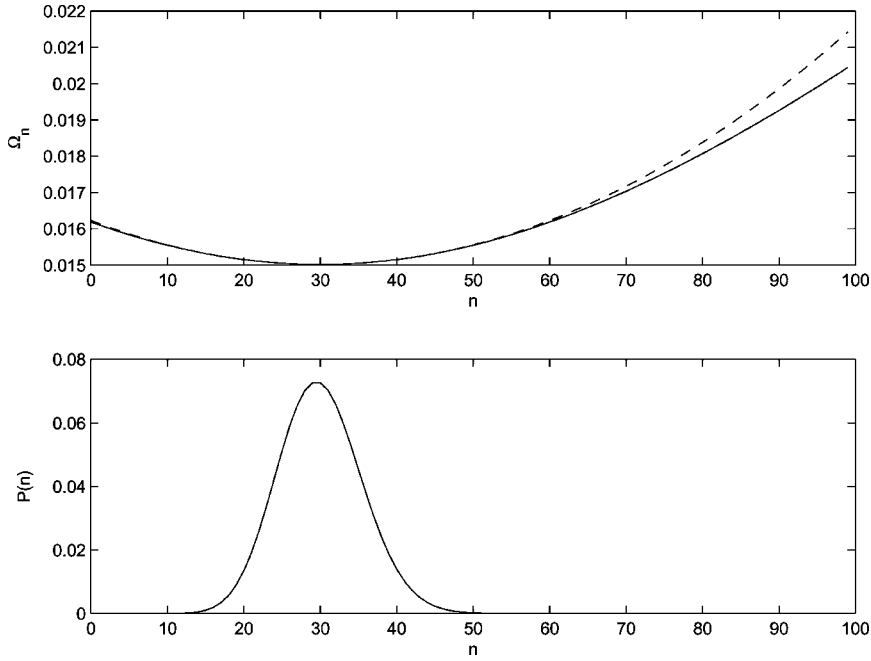

Fig. 2. Variation of Ω_n with n. We have set $g = 10^{-3}$, $k = 10^{-4}$, $\bar{n} = 30$, and $\Delta = 0.016061$. The approximate and actual Rabi frequencies are compared in the upper figure. Dotted line corresponds to the approximate expression in Eq. (32) and continuous curve corresponds to exact expression. Values of k, g, $\bar{n}$, and Δ are 10^{-4}, 10^{-3}, 30, and 0.016061, respectively. The bottom curve shows the photon number distribution for the coherent state $|\alpha = \sqrt{30}\rangle$.

In Fig. 2 the values predicted by the approximate expression for Ω_n are compared with those of the exact expression. It is clear that for the chosen values of g, k, and $\bar{n}$, the values match very well for those values of n under the peak of the photon number distribution. Quantitatively, the fractional difference is less than 3%.

3. EVOLUTION OF ATOMIC INVERSION

In the previous section we constructed the complete state of the atom–field system in the dressed atom basis in interaction picture. The state $|\phi(t)\rangle$ in the Schrödinger picture is easily obtained by premultiplying the interaction picture state function $|\psi(t)\rangle$ by $\exp(i(K_-K_+ + \nu/2)t)$. In this section, we study the temporal evolution of atomic inversion.

The time-dependent state vector $|\psi(t)\rangle$ of the system is determined completely once the coefficients $C_{e,n}(t)$ and $C_{g,n}(t)$ are known, which, in turn, are specified by their initial values. For instance, if the atom is initially in the excited

state, we have $C_{g,n}(0) = 0$ and $C_{e,n}(0) = \langle n|\psi(0)\rangle$. The probability that the atom is in the excited state, irrespective of the state of the field, is $\sum_{n=0}^{\infty} P_n \left|C_{e,n}(t)\right|^2$ and to be in the ground state is $\sum_{n=0}^{\infty} P_n \left|C_{g,n}(t)\right|^2$. The photon number distribution of the field is P_n. The difference of these two probabilities is atomic inversion. For the specified initial condition, namely, the atom is initially in the excited state, the atomic inversion $W(t)$ is

$$W(t) = 1 + \sum_{n=0}^{\infty} P_n \left[\frac{4g^2\omega^2(1+n)(1+kn)}{\Omega_n^2}(\cos(\Omega_n t) - 1)\right], \tag{33}$$

and time-dependent part of $W(t)$is

$$W_T(t) = 4g^2\omega^2 \sum_{n=0}^{\infty} P_n \frac{(1+n)(1+kn)}{\Omega_n^2} \cos(\Omega_n t). \tag{34}$$

The quantity W_T exhibits rich structure in its evolution. It exhibits collapse and revivals when the initial photon distribution is taken to be a Poissonian distribution of mean photon number $\bar{n}$, which corresponds to the field being in a coherent state $\left|\sqrt{\bar{n}}\right\rangle$. The photon number distribution for the state is $P_n = \exp(-\bar{n})\frac{\bar{n}^n}{n!}$. This distribution has a single peak and the standard deviation is $\bar{n}$. Hence, the major contribution to the sum in Eq. (34) comes from a few terms with n around the peak. With this choice of P_n, we have plotted in Fig. 3(a) the evolution of W_T as a function of time. The time required for the first collapse and the following revival, denoted by T_C and T_R, respectively, can be estimated approximately. For the revival to occur, the terms corresponding to those n around the peak, should be in phase. Thus, we require $T_R(\Omega_{\bar{n}+1} - \Omega_{\bar{n}})$ be equal to 2π. The difference of the nearest-neighbor Rabi frequencies is

$$\Omega_{\bar{n}+1} - \Omega_{\bar{n}} = \frac{2A\bar{n} + A + B}{2\Omega_{\bar{n}}}. \tag{35}$$

The constants A and B are $4(k^2\omega^2 + kg^2\omega^2)$ and $4(g^2\omega^2 + kg^2\omega^2 - \Delta k\omega)$, respectively. The revival time T_R is $\frac{4\pi\Omega_{\bar{n}}}{2A\bar{n}+A+B}$.

For the inversion to collapse, the terms in the sum on the r.h.s. of Eq. (34) should be uncorrelated. Since the width of a Possionian distribution is where the probability P_n is appreciable, the condition for collapse is written as $T_C(\Omega_{\bar{n}+\sqrt{\bar{n}}} - \Omega_{\bar{n}-\sqrt{\bar{n}}}) = 1$. If $\bar{n}$ is large, the expression for T_C is $\frac{T_R}{4\pi\sqrt{\bar{n}}}$.

The function W_T exhibits rich features when detuning is close to Δ_c. In Fig. 3. we have shown the behavior of W_T for three different values of Δ. The values for the parameters are $g = 10^{-3}$, $k = 10^{-4}$, $\omega = 1$, $\bar{n} = 30$ and the corresponding Δ_c is 0.01606. The values are so chosen that the Rabi frequency attains its minimum when n is near $\bar{n}$, the average photon number. The evolution of W_T with $\Delta = .01 < \Delta_c$ is shown as the top most figure, marked (a) in Figure 3. Figure 3(b) corresponds to $\Delta = \Delta_c$ and the one marked (c) is for $\Delta = .22 > \Delta_c$. The envelope of W_T when

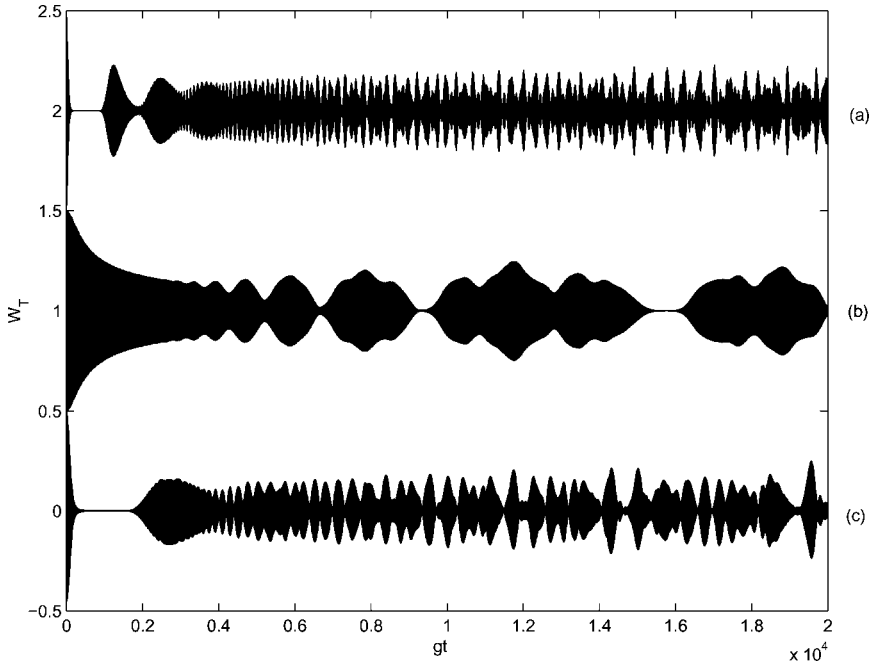

Fig. 3. Time-dependent part of atomic inversion. Case (a) corresponds to $\Delta = 0.01 < \Delta_c$. Case (b) is for $\Delta = \Delta_c = 0.016061$ and Case (c) refers to $\Delta = 0.022 > \Delta_c$. Here Case (b) shows $1 + W_T$ and Case (c) shows $2 + W_T$.

detuning equals Δ_c is distinct with structures repeating without much distortion. This should be compared with the top and bottom figures, which correspond to $\Delta \neq \Delta_c$, which exhibit random oscillations and do not have neat envelope.

The origin of these structures in the inversion is the Kerr term in the Hamiltonian and not the intensity-dependent coupling. To explain this fact, we consider the Hamiltonian $H - \chi \hat{a}^{\dagger 2}\hat{a}^2$ and the Rabi frequency is $\sqrt{\Delta + 4g^2(1 + kn)(1 + n)}$. This exhibits no minimum as Δ is independent of n, so there is no occurrence of superstructures in the evolution of atomic inversion. When $k = 1$ and Δ is zero, the inversion is composed of periodic fucntions and the results are exactly those of Buck and Sukumar (1981). For other values of k, including $k = 0$, the expression for atomic inversion involves terms of incommensurate frequencies. As a consequence the collapse and revivals are not periodic. In the case of the Hamiltonian H, the effect of k-dependent nonlinear coupling is to shift the minimum of Rabi frequency by $2k\bar{n}\omega$.

As noted in Section 2, SU(1,1) algebra is realized in terms of $K_\pm$ and K_0 when $k = 1$. If we consider resonant interaction ($r = 1$), then W_T can be estimated

approximately for the SU(1,1) case. The field is taken to be in a coherent state $|\alpha\rangle$ such that $|\alpha| = \sqrt{\bar{n}} \gg 1$. We use $(1+n)(1+kn) \approx^2$ and $g^2 \ll 1$ to arrive at $\Omega_n = 2n\omega$. With these approximations, we arrive at

$$W_T = g^2 \exp(\bar{n}(\cos(\omega t) - 1)) \cos(\bar{n} \sin(\omega t)), \tag{36}$$

in which we have used $(1+n)(1+kn) \approx \Omega_n^2$. The magnitude of W_T is negligible in this case and so there is no perceptible collapse or revival. This is due to the presence of Kerr term and the fact that it dominates over $\hat{a}^\dagger \hat{a}$ in the Hamiltonian. However, we point out that collapses and revivals are present in W_T if $\chi \ll \omega$.

4. DYNAMICS OF FIELD PROPERTIES

In the previous section, we studied the dynamics of the two-level atom, in particular, the atomic inversion. In the present section, we explore the temporal behavior of field statistics and field amplitudes. As in the previous section, the field is initially in a coherent state of complex amplitude α and the atom is taken to be in its excited state. With these initial conditions, the probability distribution of photons at time t is

$$\begin{aligned} P(n,t) &= |C_{e,n}(t)|^2 + |C_{g,n}(t)|^2 \\ &= \frac{P_n}{2}\left[1 + \frac{\Delta_n^2}{\Omega_n^2} + \frac{4g^2\omega^2(1+n)(1+kn)}{\Omega_n^2}\cos(\Omega_n t)\right] \end{aligned} \tag{37}$$

Coherent states are the wavepackets whose behavior is closest to that of a classical particle and hence are called classical states. Nevertheless, during their evolution in time the states may not be classical. The photon statistics of coherent states is Poissonian. Any deviation from this behavior is characterized by Mandel's Q parameter defined as $\frac{\langle \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} \rangle - \langle \hat{a}^\dagger \hat{a} \rangle^2}{\langle \hat{a}^\dagger \hat{a} \rangle}$. Using the time-dependent probability distribution $P(n,t)$, the expectation values in the expression for Q parameter are

$$Q = \frac{\sum_{n=0}^{\infty} n^2 P(n,t) - \left[\sum_{n=0}^{\infty} P(n,t) n\right]^2}{\sum_{n=0}^{\infty} P(n,t) n} \tag{38}$$

For coherent states, Q is unity. Any value of Q less than unity is nonclassical. In Fig. 4 the time evolution of Q parameter for an initially coherent state and $\Delta = 0.01 < \Delta_c$ is given. The emergence of nonclassical behavior ($Q < 1$) is seen. Although not shown in figure, we point out that for $\Delta = \Delta_c = 0.016061$, the statistics does not become sub-Poissonian. When $k = 1$ and $\Delta = 0$, the time-dependent part of $P(n,t)$ is of negligible magnitude and so Q does not evolve in time.

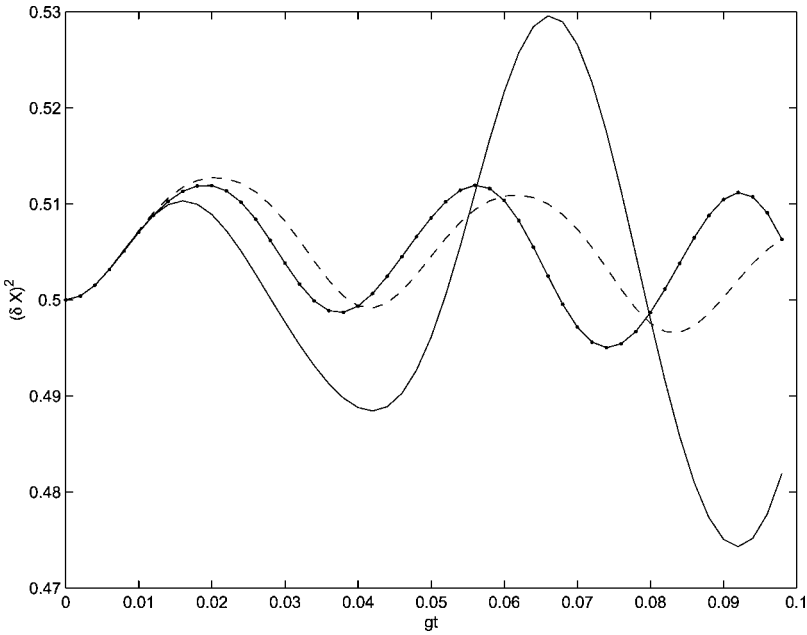

Fig. 4. Time variation of $(\delta X)^2$, as the atom–field system evolves. The evolution is shown for three values of detuning, 0.01 (continuous), 0.016061 (dotted) and 0.02 (dashed). Instants of $(\delta X)^2$ less than 0.5 correspond to squeezing in X quadrature.

4.1. Squeezing

We define the field amplitudes to be

$$X(t) = \frac{\hat{a}^\dagger(t) + \hat{a}(t)}{\sqrt{2}}, \tag{39}$$

$$Y(t) = i\frac{\hat{a}^\dagger(t) + \hat{a}(t)}{\sqrt{2}}. \tag{40}$$

These amplitudes satisfy the commutation relation $[X, Y] = i$ and hence they satisfy $(\delta X)(\delta Y) \geq 1/2$. The symbol (δX) stands for the expression $\sqrt{\langle X^2\rangle - \langle X\rangle^2}$, the variance in X for a given field state. For coherent states of any amplitude α, variances in X and Y are the same and equal to $1/\sqrt{2}$. A state is nonclassical if (δX) is less than 1/2, the coherent state value. Using the time-dependent state function given in Eq. (29), the variances in the field amplitudes are given by

$$(\delta X)^2 = \frac{1}{2}[1 + \langle 2\hat{a}^\dagger\hat{a} + \hat{a}^{\dagger 2} + \hat{a}^2\rangle - \langle\hat{a}\rangle^2 - \langle\hat{a}^\dagger\rangle^2 - 2\langle\hat{a}^\dagger\rangle\langle\hat{a}\rangle], \tag{41}$$

and

$$(\delta Y)^2 = \frac{1}{2}[1 + \langle 2\hat{a}^\dagger\hat{a} - \hat{a}^{\dagger 2} - \hat{a}^2\rangle + \langle\hat{a}\rangle^2 + \langle\hat{a}^\dagger\rangle^2 - 2\langle\hat{a}^\dagger\rangle\langle\hat{a}\rangle]. \tag{42}$$

The expectation values of various operators in these expressions are

$$\langle\hat{a}\rangle = \sum_{n=0}^{\infty}\sqrt{n+1}[C^*_{e,n}C_{e,n+1} + C^*_{g,n}C_{g,n+1}], \tag{43}$$

$$\langle\hat{a}^2\rangle = \sum_{n=0}^{\infty}\sqrt{(n+1)(1+kn)}[C^*_{e,n}C_{e,n+2} + C^*_{g,n}C_{g,n+2}], \tag{44}$$

and

$$\langle\hat{a}^\dagger\hat{a}\rangle = \sum_{n=0}^{\infty} n[C^*_{e,n}C_{e,n} + C^*_{g,n}C_{g,n}]. \tag{45}$$

The expectation values $\langle\hat{a}^\dagger\rangle$ and $\langle\hat{a}^{\dagger 2}\rangle$ are the complex conjugates of $\langle\hat{a}\rangle$ and $\langle\hat{a}^2\rangle$, respectively.

The evolution of δX is shown in Fig. 5. As the field evolves in time, the variance in X falls below 0.5 indicating that the quadrature exhibits squeezing. As

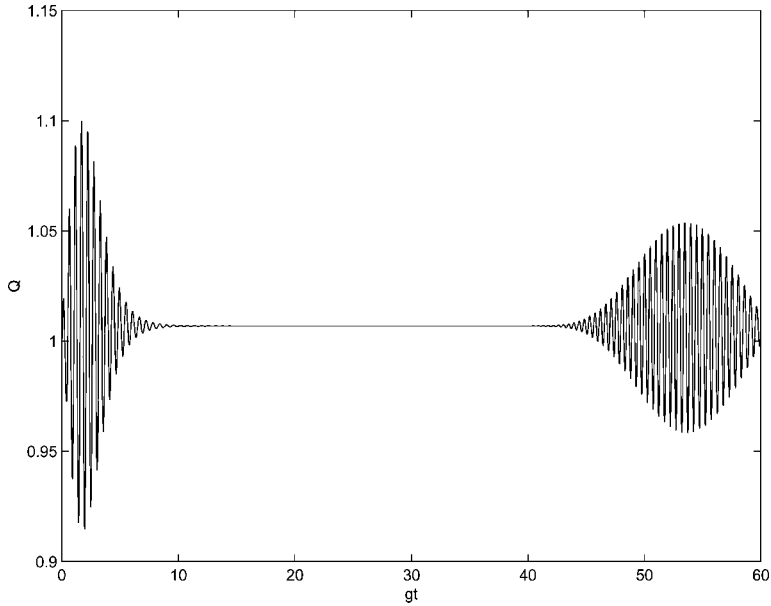

Fig. 5. Mandel's Q parameter as a function of time. Instants of $Q < 1$ correspond to sub-Poissonian statistics. Values of g, k and $\bar{n}$ are same as in Fig. 2 and $\Delta = 0.01$.

a consequence of uncertainty relation, the Y quadrature does not show squeezing. However, when we set $\alpha = i\sqrt{30}$, the situation is reversed. In this case, squeezing is possible in Y and not in X.

4.2. Overlap of Initial and Time-Evolved States

A quantity of interest is the overlap of the state of the atom–field at time t and that at $t = 0$. With the same initial conditions for the atom–field state as in the previous section, the overlap is

$$|\langle\psi(0)|\psi(t)\rangle|^2 = \exp(-|\alpha|^2)\left|\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!}\left[\cos\left(\frac{\Omega_n t}{2}\right) + i\frac{\Delta_n}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right)\right]\right|^2. \tag{46}$$

The numerical value of the overlap lies between zero and unity. It is seen from Fig. 6 that the overlap becomes zero at longer times. In other words, the time-evolved state is almost orthogonal to initial state. For short durations, an approximate expression

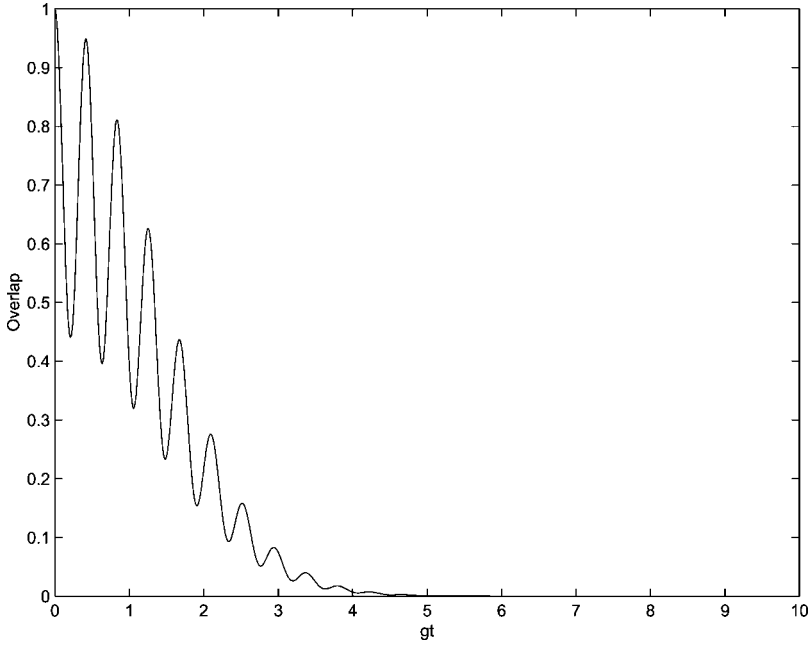

Fig. 6. Overlap of initial and time-evolved field states. Y-axis corresponds to $|\langle\psi(0)|\psi(t)\rangle|^2$. The envelope of the overlap function decays with time implying that the initial state is almost orthogonal to the evolved state. The values of the parameters are same as in Fig. 2 and the detuning Δ is equal to 0.016061.

for the decay of overlap is derived by replacing expression $\exp(-\bar{n})\frac{\bar{n}^n}{n!}$ with the Gaussian distribution $\exp(-\frac{(n-\bar{n})^2}{2\bar{n}})$ and the sum over n by integration. Further, we set $\Omega_n = \Omega_N + (N-n)\Omega'_N + \Omega''_N(N-n)^2$, with the assumption that N is close to but not less than $\bar{n}$. Here, prime denotes taking derivative with respect to n and the suffix represents the value of n where the derivative is evaluated. With these approximations we get, after neglecting oscillatory terms,

$$\sum_{n=0}^{\infty} \exp\left(-\frac{(n-\bar{n})^2}{2\bar{n}}\right)\cos\left(\frac{\Omega_n t}{2}\right) = Re\int_0^{\infty} \exp\left(-\frac{(n-\bar{n})^2}{2\bar{n}} + i\frac{\Omega_n t}{2}\right) dn$$

$$\propto \left[1 + N^2\frac{\partial^2\Omega_n}{\partial^2 n}|_{n=N}\frac{t^2}{4}\right]^{-\frac{1}{4}} \times \exp\left[-N^2\frac{\partial\Omega_n}{\partial n}|_{n=N}\frac{t^2}{4}\right] \tag{47}$$

Similar expression can be derived for summation with $\sin(\frac{\Omega_n t}{2})$. The above expression indeed predicts that the overlap function decays with time. When $N = \bar{n}$, the first derivative of Ω_n vanishes and the exponential term in the envelope is absent. Consequently the decay is slower. However, if the photon number distribution of the field is very broad, it is incorrect to truncate the Taylor series and the expression in Eq. (47) is invalid.

5. SUMMARY

The Hamiltonian that contains the usual and intensity-dependent (including Kerr term) JC models as limiting cases has been constructed. The importance of the model is that the nondissipative dynamics dictated by the generalized Hamiltonian H is completely solvable. The model approximates in various limiting cases many of the well known and realizable Hamiltonians. The algebra relavant to the model has been shown to be SU(1,1) or Heisenberg—Weyl algebra depending on whether k is unity or zero. The eigenvalues of the Hamiltonian exhibits level repulsion. In the case of nonvanishing Kerr term, the Rabi frequency, which is a function of the photon number n, attains a minimum. The dynamical behavior of atomic inversion, when detuning is so chosen so that the Rabi frequency attains its minimum, exhibits superstructures, which are absent in the usual JC model. The overlap of the initial coherent state and the time-evolved state decays with time. In the language of inner product, the initial coherent state of the field is almost orthogonal to its time-evolved state. The expressions derived in the paper go over to those of the JC model if $k \to 0$.

We note that the formal equivalence between $H_{\rm JC}$ and H is obtained by identifying K_+ with $\bar{a}^\dagger$ and K_- with $\bar{a}$. This, in conjunction with the fact both the sets of operators $\{\hat{a}^\dagger, \hat{a}, I\}$ and $\{K_\pm, K_0\}$ are closed under commutation, implies that the expansion coefficients $C_{e,n}(t)$ and $C_{g,n}(t)$ for the evolution governed by H are obtained from the corresponding expressions for the usual JC model by

replacing Δ by Δ_n and g by $g\sqrt{1+kn}$. Hence, all those physical quantities, like the atomic inversion, quadrature fluctuations, etc., computed in terms of the expansion coefficients are derivable from those of the JC model.

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